Published by Institute of Physics Publishing for SISSA

Received: *October 8, 2007*
Revised: *January 2, 2008*
Accepted: *February 21, 2008*
Published: *March 3, 2008*

JHEP03(2008)001

Integrablility of a classical $N = 2$ super sinh-Gordon model with jump defects

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Abstract: The Lagrangian formalism for the $N = 2$ supersymmetric sinh-Gordon model with a jump defect is considered. The modified conserved momentum and energy are constructed in terms of border functions. The supersymmetric Backlund transformation is given and an one-soliton solution is obtained.
The Lax formulation based on the affine super Lie algebra $sl(2, 2)$ within the space split by the defect leads to the integrability of the model and henceforth to the existence of an infinite number of constants of motion.

Keywords: Integrable Hierarchies, Integrable Field Theories.

Contents

1. Introduction

In reference [1] a quantum theory of free bosons and free fermions subject to an internal boundary condition (jump defect) preserving the integrability was considered. More recently a classical Lagrangian approach was proposed for certain class of non linearly interacting bosonic fields [2−4]. The authors have considered a class of systems described by internal boundary conditions corresponding to Backlund transformations and showed to preserve integrability. Such framework was generalized in [5] to include both bosons and fermions interacting in a non linear manner. Specifically we have considered the super sinh-Gordon model with $N = 1$. The border functions were constructed and shown to give rise to Backlund transformations. The integrability of the system in the presence of this kind of defect was considered in terms of zero curvature representation.

The $N = 2$ super sinh-Gordon model was proposed in [6, 7] using superfield formalism. Much later, a systematic construction was developed from the algebraic formalism point of view where the $N = 2$ super sinh-Gordon equations appears to be a member of an integrable hierarchy [8, 9]. The description of integrability in terms of an affine algebraic structure has provided a neat universal framework from which the dynamics, conservation laws, soliton solutions, etc can be constructed and studied. Following the same line of reasoning, supersymmetry transformation was also incorporated into the framework (see e.g. [8, 9]) associated to half integer gradations.

The purpose of this paper is to extend the results obtained in [5] for the $N = 2$ super sinh-Gordon model. In section 2 we discuss the Lagrangian formalism introducing the jump defect in terms of the border functions. These are defined by a modified momentum conservation which is consistent with the Backlund transfomation for the $N = 2$ super sinh-Gordon. The presence of the defect require modification of the conserved energy of the system. In section 3 we extend the construction of Backlund transformation of ref. [10] in terms of super fields to the $N = 2$ super sinh-Gordon. We also write down the the Backlund transformation in components and obtain a one-soliton solution. These are invariant under the $N = 2$ supersymmetry transformation. In section 4 we present the zero curvature formulation in terms of an affine $sl(2, 2)$ super Lie algebra. By introducing two regions around the defect, we explicitly construct, in a closed form, a gauge group element connection the Lax pair in the overlap region. This fact guarantees the existence of an infinite set of conservation laws.

2. Lagrangian description

The starting point is the Lagrangian density describing the $N = 2$ super sinh-Gordon theory with bosonic ϕ_1, φ_1 and fermionic $\bar{\psi}_1, \psi_1, \bar{\chi}_1, \chi_1$ fields in the region $x < 0$ and corresponding ϕ_2, φ_2 and $\bar{\psi}_2, \psi_2, \bar{\chi}_2, \chi_2$ for $x > 0$,

$$\begin{aligned}\mathcal{L}_p = &\frac{1}{2}(\partial_x\phi_p)^2 - \frac{1}{2}(\partial_t\phi_p)^2 + 2\bar{\psi}_p\partial_t\bar{\psi}_p + 2\bar{\psi}_p\partial_x\bar{\psi}_p + 2\psi_p\partial_t\psi_p - 2\psi_p\partial_x\psi_p \\ &-\frac{1}{2}(\partial_x\varphi_p)^2 + \frac{1}{2}(\partial_t\varphi_p)^2 - 2\bar{\chi}_p\partial_t\bar{\chi}_p - 2\bar{\chi}_p\partial_x\bar{\chi}_p - 2\chi_p\partial_t\chi_p + 2\chi_p\partial_x\chi_p \\ &-16(\psi_p\bar{\psi}_p + \chi_p\bar{\chi}_p)\cosh\varphi_p\cosh\phi_p - 4\cosh(2\varphi_p) \\ &+16(\psi_p\bar{\chi}_p + \chi_p\bar{\psi}_p)\sinh\varphi_p\sinh\phi_p + 4\cosh(2\phi_p), \qquad p = 1, 2. \end{aligned} \tag{2.1}$$

We shall now consider the system with a defect at the origin ($x = 0$) described by

$$\mathcal{L} = \theta(-x)\mathcal{L}_1 + \theta(x)\mathcal{L}_2 + \delta(x)\mathcal{L}_D \tag{2.2}$$

where

$$\begin{aligned}\mathcal{L}_D = &1/2(\phi_2\partial_t\phi_1 - \phi_1\partial_t\phi_2) - 2\psi_1\psi_2 - 2\bar{\psi}_1\bar{\psi}_2 + \zeta_1^-\partial_t\zeta_1^+ \\ &-1/2(\varphi_2\partial_t\varphi_1 - \varphi_1\partial_t\varphi_2) + 2\chi_1\chi_2 + 2\bar{\chi}_1\bar{\chi}_2 + \zeta_1^+\partial_t\zeta_1^- \\ &+B_0(\phi_p, \varphi_p) + B_1(\phi_p, \varphi_p, \psi_p, \chi_p, \bar{\psi}_p, \bar{\chi}_p, \zeta_1^-, \zeta_1^+). \end{aligned} \tag{2.3}$$

The boundary functions $B = B_0 + B_1$ describe the defect and $\zeta_1^{\pm}$ are fermionic auxiliary fields. Notice that they describe the effect of the jump defect on the fermionic fields generalizing the $N = 1$ case [5] where one auxiliary field appears.

The equations of motion are

$$\begin{aligned}(\partial_x^2 - \partial_t^2)\phi_p &= 8\text{sinh}(2\phi_p) - 16(\psi_p\bar{\psi}_p + \chi_p\bar{\chi}_p)\text{sinh}\phi_p\text{cosh}\varphi_p \\ &\quad +16(\psi_p\bar{\chi}_p + \chi_p\bar{\psi}_p)\text{sinh}\varphi_p\text{cosh}\phi_p \\ (\partial_x^2 - \partial_t^2)\varphi_p &= 8\text{sinh}(2\varphi_p) + 16(\psi_p\bar{\psi}_p + \chi_p\bar{\chi}_p)\text{sinh}\varphi_p\text{cosh}\phi_p \\ &\quad -16(\psi_p\bar{\chi}_p + \chi_p\bar{\psi}_p)\text{sinh}\phi_p\text{cosh}\varphi_p \\ (\partial_x - \partial_t)\psi_p &= -4\bar{\psi}_p\text{cosh}\phi_p\text{cosh}\varphi_p + 4\bar{\chi}_p\text{sinh}\phi_p\text{sinh}\varphi_p \\ (\partial_x - \partial_t)\chi_p &= -4\bar{\psi}_p\text{sinh}\phi_p\text{sinh}\varphi_p + 4\bar{\chi}_p\text{cosh}\phi_p\text{cosh}\varphi_p \\ (\partial_x + \partial_t)\bar{\psi}_p &= -4\psi_p\text{cosh}\phi_p\text{cosh}\varphi_p + 4\chi_p\text{sinh}\phi_p\text{sinh}\varphi_p \end{aligned}$$

$$(\partial_x + \partial_t)\bar{\chi}_p \;=\; -4\psi_p\text{sinh}\phi_p\text{sinh}\varphi_p + 4\chi_p\text{cosh}\phi_p\text{cosh}\varphi_p \tag{2.4}$$

where $p = 1, 2$ corresponds to $x < 0$ or $x > 0$ respectively. These equations are invariant under the supersymmetry transformation,

$$\begin{aligned}
\delta(\phi_p \pm \varphi_p) &= 2(\psi_p \mp \chi_p)\epsilon_\pm, \\
\delta(\psi_p \pm \chi_p) &= -1/2(\partial_x + \partial_t)(\phi_p \mp \varphi_p)\epsilon_\pm, \\
\delta(\bar{\psi}_p \pm \bar{\chi}_p) &= 2\sinh(\phi_p \pm \varphi_p)\epsilon_\mp.
\end{aligned} \tag{2.5}$$

where $\epsilon_\pm$ are fermionic parameters. In $x = 0$ we have,

$$\begin{aligned}
\partial_x\phi_1 - \partial_t\phi_2 &= -\partial_{\phi_1}B, & \partial_x\phi_2 - \partial_t\phi_1 &= \partial_{\phi_2}B, \\
\partial_x\varphi_1 - \partial_t\varphi_2 &= \partial_{\varphi_1}B, & \partial_x\varphi_2 - \partial_t\varphi_1 &= -\partial_{\varphi_2}B, \\
\psi_1 - \psi_2 &= -\frac{1}{2}\partial_{\psi_1}B = -\frac{1}{2}\partial_{\psi_2}B & \chi_1 - \chi_2 &= \frac{1}{2}\partial_{\chi_1}B = \frac{1}{2}\partial_{\chi_2}B \\
\bar{\psi}_1 + \bar{\psi}_2 &= \frac{1}{2}\partial_{\bar{\psi}_1}B = -\frac{1}{2}\partial_{\bar{\psi}_2}B, & \bar{\chi}_1 + \bar{\chi}_2 &= -\frac{1}{2}\partial_{\bar{\chi}_1}B = \frac{1}{2}\partial_{\bar{\chi}_2}B, \\
\partial_t\zeta_1^- &= -\frac{1}{2}\partial_{\zeta_1^+}B, & \partial_t\zeta_1^+ &= -\frac{1}{2}\partial_{\zeta_1^-}B
\end{aligned} \tag{2.6}$$

The canonical momentum P, given by

$$\begin{aligned}
P = &\int_{-\infty}^{0} \left(\partial_x\phi_1\partial_t\phi_1 - 2\bar{\psi}_1\partial_x\bar{\psi}_1 - 2\psi_1\partial_x\psi_1 \,-\, \partial_x\varphi_1\partial_t\varphi_1 + 2\bar{\chi}_1\partial_x\bar{\chi}_1 + 2\chi_1\partial_x\chi_1\right) dx \\
&+ \int_{0}^{\infty} \left(\partial_x\phi_2\partial_t\phi_2 - 2\bar{\psi}_2\partial_x\bar{\psi}_2 - 2\psi_2\partial_x\psi_2 \,-\, \partial_x\varphi_2\partial_t\varphi_2 + 2\bar{\chi}_2\partial_x\bar{\chi}_2 + 2\chi_2\partial_x\chi_2\right) dx
\end{aligned} \tag{2.7}$$

is not conserved in time due to the presence of the defect. Instead we find after making use of the equations of motion,

$$\begin{aligned}
\dot{P} = &\Bigg[\frac{1}{2}(\partial_t\phi_1)^2 - \frac{1}{2}(\partial_t\varphi_1)^2 + \frac{1}{2}(\partial_x\phi_1)^2 - \frac{1}{2}(\partial_x\varphi_1)^2 - 2\bar{\psi}_1\partial_t\bar{\psi}_1 - 2\psi_1\partial_t\psi_1 \\
&+2\bar{\chi}_1\partial_t\bar{\chi}_1 + 2\chi_1\partial_t\chi_1 - 4\text{cosh}(2\phi_1) + 4\text{cosh}(2\varphi_1) + 16(\psi_1\bar{\psi}_1 + \chi_1\bar{\chi}_1)\text{cosh}\phi_1\text{cosh}\varphi_1 \\
&-16(\psi_1\bar{\chi}_1 + \chi_1\bar{\psi}_1)\text{sinh}\phi_1\text{sinh}\varphi_1\Bigg]_{x=0} - \Bigg[(1 \to 2)\Bigg]_{x=0}
\end{aligned}$$

Factorizing $B_0 = B_0^{(+)} + B_0^{(-)}$ and $B_1 = B_1^{(+)} + B_1^{(-)}$, where $\phi_\pm = \phi_1 \pm \phi_2$, $\varphi_\pm = \varphi_1 \pm \varphi_2, \cdots$ such that

$$\begin{aligned}
B_0^{(+)} &= B_0^{(+)}(\phi_+, \varphi_+), & B_0^{(-)} &= B_0^{(-)}(\phi_-, \varphi_-) \\
B_1^{(+)} &= B_1^{(+)}(\phi_+, \varphi_+, \psi_+, \chi_+, \zeta_1^-, \zeta_1^+), & B_1^{(-)} &= B_1^{(-)}(\phi_-, \varphi_-, \bar{\psi}_-, \bar{\chi}_-, \zeta_1^-, \zeta_1^+)
\end{aligned} \tag{2.8}$$

and using eqs. (2.6), we define the modified momentum

$$\mathcal{P} = P + \left[B_0^{(+)} - B_0^{(-)} + B_1^{(+)} - B_1^{(-)} + 2\bar{\psi}_1\bar{\psi}_2 - 2\psi_1\psi_2 - 2\bar{\chi}_1\bar{\chi}_2 + 2\chi_1\chi_2\right]|_{x=0} \tag{2.9}$$

which is conserved in time provided the border functions B_0 and B_1 satisfy

$$\partial_{\phi_+} B_0^{(+)} \partial_{\phi_-} B_0^{(-)} - \partial_{\varphi_+} B_0^{(+)} \partial_{\varphi_-} B_0^{(-)} = 4\sinh\phi_+ \sinh\phi_- - 4\sinh\varphi_+ \sinh\varphi_- \quad (2.10)$$

and

$$\begin{aligned}
&\partial_{\phi_+} B_0^{(+)} \partial_{\phi_-} B_1^{(-)} + \partial_{\phi_-} B_0^{(-)} \partial_{\phi_+} B_1^{(+)} - \partial_{\varphi_+} B_0^{(+)} \partial_{\varphi_-} B_1^{(-)} - \partial_{\varphi_-} B_0^{(-)} \partial_{\varphi_+} B_1^{(+)} \\
&+\partial_{\phi_+} B_1^{(+)} \partial_{\phi_-} B_1^{(-)} - \partial_{\varphi_+} B_1^{(+)} \partial_{\varphi_-} B_1^{(-)} - \frac{1}{2}(\partial_{\zeta_1^-} B_1^{(-)} \partial_{\zeta_1^+} B_1^{(+)} + \partial_{\zeta_1^+} B_1^{(-)} \partial_{\zeta_1^-} B_1^{(+)}) \\
&\quad = -2(\psi_+\bar{\psi}_+ + \psi_-\bar{\psi}_- + \chi_+\bar{\chi}_+ + \chi_-\bar{\chi}_-)\Lambda_- - 2(\psi_+\bar{\psi}_- + \psi_-\bar{\psi}_+ + \chi_+\bar{\chi}_- + \chi_-\bar{\chi}_+)\Lambda_+ \\
&\qquad +2(\psi_+\bar{\chi}_+ + \psi_-\bar{\chi}_- + \chi_+\bar{\psi}_+ + \chi_-\bar{\psi}_-)\Delta_- + 2(\psi_+\bar{\chi}_- + \psi_-\bar{\chi}_+ + \chi_+\bar{\psi}_- + \chi_-\bar{\psi}_+)\Delta_+
\end{aligned} \quad (2.11)$$

where

$$\begin{aligned}
\Lambda_\pm &= \cosh\left(\frac{\phi_+ + \phi_-}{2}\right)\cosh\left(\frac{\varphi_+ + \varphi_-}{2}\right) \pm \cosh\left(\frac{\phi_+ - \phi_-}{2}\right)\cosh\left(\frac{\varphi_+ - \varphi_-}{2}\right), \\
\Delta_\pm &= \sinh\left(\frac{\phi_+ + \phi_-}{2}\right)\sinh\left(\frac{\varphi_+ + \varphi_-}{2}\right) \pm \sinh\left(\frac{\phi_+ - \phi_-}{2}\right)\sinh\left(\frac{\varphi_+ - \varphi_-}{2}\right),
\end{aligned} \quad (2.12)$$

The energy of the system with the defect is given by

$$E = \int_{-\infty}^{0} dx\, \mathcal{H}_1 + \int_0^\infty dx\, \mathcal{H}_2,$$

where

$$\begin{aligned}
\mathcal{H}_p = &\left[\frac{1}{2}(\partial_x\phi_p)^2 + \frac{1}{2}(\partial_t\phi_p)^2 - \frac{1}{2}(\partial_x\varphi_p)^2 - \frac{1}{2}(\partial_t\varphi_p)^2 - 2\psi_p\partial_x\psi_p + 2\bar{\psi}_p\partial_x\bar{\psi}_p \right. \quad (2.13)\\
&+2\chi_p\partial_x\chi_p - 2\bar{\chi}_p\partial_x\bar{\chi}_p - 16(\psi_p\bar{\psi}_p + \chi_p\bar{\chi}_p)\cosh\varphi_p \cosh\phi_p \\
&\left. +16(\psi_p\bar{\chi}_p + \chi_p\bar{\psi}_p)\sinh\varphi_p \sinh\phi_p + 4\cosh(2\phi_p) - 4\cosh(2\varphi_p)\right], \quad p = 1, 2
\end{aligned}$$

It follows after using the equations of motion (2.4) that

$$\begin{aligned}
\frac{dE}{dt} = &\left[\partial_x\phi_1\partial_t\phi_1 - \partial_x\varphi_1\partial_t\varphi_1 - 2\psi_1\partial_t\psi_1 + 2\bar{\psi}_1\partial_t\bar{\psi}_1 + 2\chi_1\partial_t\chi_1 - 2\bar{\chi}_1\partial_t\bar{\chi}_1\right]_{x=0} \\
&- \left[\partial_x\phi_2\partial_t\phi_2 - \partial_x\varphi_2\partial_t\varphi_2 - 2\psi_2\partial_t\psi_2 + 2\bar{\psi}_2\partial_t\bar{\psi}_2 + 2\chi_2\partial_t\chi_2 - 2\bar{\chi}_2\partial_t\bar{\chi}_2\right]_{x=0}.
\end{aligned}$$

Inserting the Backlund transformation (2.6) we can define the modified conserved energy,

$$\mathcal{E} = E + [B - 2\psi_1\psi_2 + 2\chi_1\chi_2 - 2\bar{\psi}_1\bar{\psi}_2 + 2\bar{\chi}_1\bar{\chi}_2]|_{x=0}, \quad (2.14)$$

where $B = B_0^{(+)} + B_0^{(-)} + B_1^{(+)} + B_1^{(-)}$.

Equation (2.10) has the following solution

$$\begin{aligned}
B_0^{(+)} &= B_0^{(+)}(\phi_+, \varphi_+) = \frac{2\beta_3}{\beta_2}(\cosh\phi_+ - \cosh\varphi_+), \\
B_0^{(-)} &= B_0^{(-)}(\phi_-, \varphi_-) = \frac{2\beta_2}{\beta_3}(\cosh\phi_- - \cosh\varphi_-).
\end{aligned} \quad (2.15)$$

The solution of (2.11) is verified for

$$
\begin{aligned}
B_1^{(+)} &= \frac{i}{\sqrt{2}}\zeta_1^- \left(-\beta_3(\psi_+ - \chi_+)\cosh\frac{1}{2}(\phi_+ + \varphi_+)\right) \\
&\quad + \frac{i}{\sqrt{2}}\zeta_1^+ \frac{\beta_1}{\beta_3}\left(-\beta_3(\psi_+ + \chi_+)\cosh\frac{1}{2}(\phi_+ - \varphi_+)\right), \qquad (2.16)\\
B_1^{(-)} &= \frac{i}{\sqrt{2}}\zeta_1^- \left(\beta_2(\bar{\psi}_- + \bar{\chi}_-)\cosh\frac{1}{2}(\phi_- - \varphi_-)\right) \\
&\quad + \frac{i}{\sqrt{2}}\zeta_1^+ \frac{\beta_1}{\beta_3}\left(\beta_2(\bar{\psi}_- - \bar{\chi}_-)\cosh\frac{1}{2}(\phi_- + \varphi_-)\right) \qquad (2.17)
\end{aligned}
$$

where β_1, β_2 and β_3 are arbitrary constants.

The solution given above also verify the following identities

$$
\begin{aligned}
&(\partial_{\phi_+} B_0^{(+)}\partial_{\phi_-} B_1^{(-)} + \partial_{\phi_-} B_0^{(-)}\partial_{\phi_+} B_1^{(+)}) - (\partial_{\varphi_+} B_0^{(+)}\partial_{\varphi_-} B_1^{(-)} + \partial_{\varphi_-} B_0^{(-)}\partial_{\varphi_+} B_1^{(+)}) \qquad (2.18)\\
&\quad = -2(\psi_+\bar{\psi}_+ + \psi_-\bar{\psi}_- + \chi_+\bar{\chi}_+ + \chi_-\bar{\chi}_-)\Lambda_- + 2(\psi_+\bar{\chi}_+ + \psi_-\bar{\chi}_- + \chi_+\bar{\psi}_+ + \chi_-\bar{\psi}_-)\Delta_-\\
&\frac{1}{2}(\partial_{\zeta_1^-} B_1^{(-)}\partial_{\zeta_1^+} B_1^{(+)} + \partial_{\zeta_1^+} B_1^{(-)}\partial_{\zeta_1^-} B_1^{(+)}) = 2(\psi_+\bar{\psi}_- + \chi_+\bar{\chi}_-)\Lambda_+ - 2(\psi_+\bar{\chi}_- + \chi_+\bar{\psi}_-)\Delta_+ \qquad (2.19)\\
&(\partial_{\phi_+} B_1^{(+)}\partial_{\phi_-} B_1^{(-)} - \partial_{\varphi_+} B_1^{(+)}\partial_{\varphi_-} B_1^{(-)}) = 0 \qquad (2.20)\\
&(\psi_-\bar{\psi}_+ + \chi_-\bar{\chi}_+)\Lambda_+ - (\psi_-\bar{\chi}_+ + \chi_-\bar{\psi}_+)\Delta_+ = 0 \qquad (2.21)
\end{aligned}
$$

In analogy with ref. [5] where we have dealt with the $N = 1$ case, the space derivatives of $\zeta_1^\pm$, i.e. $\partial_x \zeta_1^\pm$ can be obtained by requiring compatibility of eqs. (2.6) with (2.4) with B given by (2.15)–(2.17). Explicit expressions are given in (3.25) and (3.26) in consistency with the Backlund transformation where the auxiliary fields $\zeta_1^\pm$ appear in a natural manner.

3. Backlund transformation — One-soliton solution

Introducing

$$
\partial_z = \frac{1}{2}(\partial_x + \partial_t), \quad \partial_{\bar{z}} = \frac{1}{2}(\partial_t - \partial_x), \qquad z = x + t, \quad \bar{z} = t - x \qquad (3.1)
$$

we define, according to ref. [7], the super fields

$$
\begin{aligned}
\Upsilon^+ &= \eta^+(z^+, \bar{z}^+) + \theta^+\psi^-(z^+, \bar{z}^+) + \bar{\theta}^+\bar{\psi}^-(z^+, \bar{z}^+) + \theta^+\bar{\theta}^+F^+(z^+, \bar{z}^+)\\
\Upsilon^- &= \eta^-(z^-, \bar{z}^-) + \theta^-\psi^+(z^-, \bar{z}^-) + \bar{\theta}^-\bar{\psi}^+(z^-, \bar{z}^-) + \theta^-\bar{\theta}^-F^-(z^-, \bar{z}^-)
\end{aligned}
$$

where

$$
z^\pm = z \pm \frac{1}{2}\theta^+\theta^- \qquad \bar{z}^\pm = \bar{z} \pm \frac{1}{2}\bar{\theta}^+\bar{\theta}^-
$$

and the super derivatives

$$
\begin{aligned}
D_+ &= \frac{\partial}{\partial\theta^+} + \frac{1}{2}\theta^-\partial_z, \qquad \bar{D}_+ = \frac{\partial}{\partial\bar{\theta}^+} + \frac{1}{2}\bar{\theta}^-\partial_{\bar{z}},\\
D_- &= \frac{\partial}{\partial\theta^-} + \frac{1}{2}\theta^+\partial_z, \qquad \bar{D}_- = \frac{\partial}{\partial\bar{\theta}^-} + \frac{1}{2}\bar{\theta}^+\partial_{\bar{z}},
\end{aligned}
$$

The equations of motion are then given by [7]

$$\bar{D}_+ D_+ \Upsilon^+ = g\sin\Upsilon^-, \qquad \bar{D}_- D_- \Upsilon^- = g\sin\Upsilon^+ \tag{3.2}$$

where g is a constant. In components we find for the first equation (3.2)

$$\begin{aligned} F^+ &= g\sin\eta^- & \partial_{\bar{z}}\psi^- &= g\cos\eta^-\bar{\psi}^+ \\ \partial_z\bar{\psi}^- &= -g\cos\eta^-\psi^+ & \partial_z\partial_{\bar{z}}\eta^+ &= -g\cos\eta^- F^- - g\sin\eta^-\psi^+\bar{\psi}^+ \end{aligned}$$

while for the second equation (3.2) we have

$$\begin{aligned} F^- &= g\sin\eta^+ & \partial_{\bar{z}}\psi^+ &= g\cos\eta^+\bar{\psi}^- \\ \partial_z\bar{\psi}^+ &= -g\cos\eta^+\psi^- & \partial_z\partial_{\bar{z}}\eta^- &= -g\cos\eta^+ F^+ - g\sin\eta^+\psi^-\bar{\psi}^- \end{aligned}$$

The superfields Υ^+ e Υ^- are chiral and satisfy

$$\bar{D}_+\Upsilon^- = D_+\Upsilon^- = 0, \qquad \bar{D}_-\Upsilon^+ = D_-\Upsilon^+ = 0 \tag{3.3}$$

Extending the procedure given in [10], we propose for the first eq. (3.2) the following Backund transformation

$$D_+\Upsilon_1^+ = D_+\Upsilon_2^+ + \beta_1\mathcal{F}_1\cos\left(\frac{\Upsilon_1^- + \Upsilon_2^-}{2}\right) \tag{3.4}$$

$$\bar{D}_+\Upsilon_1^+ = -\bar{D}_+\Upsilon_2^+ + \beta_2\mathcal{F}_2\cos\left(\frac{\Upsilon_1^- - \Upsilon_2^-}{2}\right) \tag{3.5}$$

where $\mathcal{F}_1$ and $\mathcal{F}_2$ are auxiliary fermionic superfields, β_1 and β_2 are arbitrary constants. From

$$(\bar{D}_+ D_+ + D_+\bar{D}_+)\Upsilon_1^+ = 0 \tag{3.6}$$

we obtain

$$\bar{D}_+ D_+\Upsilon_2^+ = g\sin\Upsilon_2^- \tag{3.7}$$

if the auxiliary superfields $\mathcal{F}_1$ and $\mathcal{F}_2$ satisfy

$$\bar{D}_+\mathcal{F}_1 = \frac{2g}{\beta_1}\sin\left(\frac{\Upsilon_1^- - \Upsilon_2^-}{2}\right) \qquad D_+\mathcal{F}_2 = -\frac{2g}{\beta_2}\sin\left(\frac{\Upsilon_1^- + \Upsilon_2^-}{2}\right) \tag{3.8}$$

For the second equation (3.2) consider the following transformations

$$D_-\Upsilon_1^- = D_-\Upsilon_2^- + \beta_3\mathcal{G}_1\cos\left(\frac{\Upsilon_1^+ + \Upsilon_2^+}{2}\right) \tag{3.9}$$

$$\bar{D}_-\Upsilon_1^- = -\bar{D}_-\Upsilon_2^- + \beta_4\mathcal{G}_2\cos\left(\frac{\Upsilon_1^+ - \Upsilon_2^+}{2}\right) \tag{3.10}$$

where $\mathcal{G}_1$ and $\mathcal{G}_2$ are auxiliary fermionic superfields and β_3 and β_4 are arbitrary constants. Similarly, from

$$(\bar{D}_- D_- + D_-\bar{D}_-)\Upsilon_1^- = 0 \tag{3.11}$$

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we obtain the second equation

$$\bar{D}_- D_- \Upsilon_2^- = g \sin \Upsilon_2^+$$

provided $\mathcal{G}_1$ and $\mathcal{G}_2$ satisfy the following conditions

$$\bar{D}_- \mathcal{G}_1 = \frac{2g}{\beta_3} \sin\left(\frac{\Upsilon_1^+ - \Upsilon_2^+}{2}\right) \qquad D_- \mathcal{G}_2 = -\frac{2g}{\beta_4} \sin\left(\frac{\Upsilon_1^+ + \Upsilon_2^+}{2}\right) \tag{3.12}$$

In the appendix A we derive further compatibility relations involving the Fermionic auxiliary superfields. These are written explicitly in components and provide algebraic relations like (A.7) also.

Choosing $g = 2$, redefining fields

$$\eta_p^\pm \to i(\phi_p \pm \varphi_p), \quad \psi_p^\pm \to i\sqrt{2}(\psi_p \pm \chi_p), \quad \bar{\psi}_p^\pm \to i\sqrt{2}(\bar{\psi}_p \pm \bar{\chi}_p), \quad p = 1, 2. \tag{3.13}$$

and denoting

$$\Phi_s^{(\pm)} = \phi_s \pm \varphi_s, \quad \Psi_s^{(\pm)} = \psi_s \pm \chi_s, \quad \bar{\Psi}_s^{(\pm)} = \bar{\psi}_s \pm \bar{\chi}_s, \quad s = \pm \tag{3.14}$$

where $\phi_\pm = \phi_1 \pm \phi_2, \quad \varphi_\pm = \varphi_1 \pm \varphi_2, \quad \psi_\pm = \psi_1 \pm \psi_2, \quad \chi_\pm = \chi_1 \pm \chi_2, \cdots$ In components the Backlund transformation reads

$$\begin{aligned}
\partial_x \phi_1 - \partial_t \phi_2 = &\frac{i}{2\sqrt{2}} \zeta_1^- \left[-\beta_2 \sinh\left(\frac{\Phi_-^{(-)}}{2}\right) \bar{\Psi}_-^{(+)} + \beta_3 \sinh\left(\frac{\Phi_+^{(+)}}{2}\right) \Psi_+^{(-)} \right] \\
&+ \frac{i}{2\sqrt{2}} \frac{\beta_1}{\beta_3} \zeta_1^+ \left[-\beta_2 \sinh\left(\frac{\Phi_-^{(+)}}{2}\right) \bar{\Psi}_-^{(-)} + \beta_3 \sinh\left(\frac{\Phi_+^{(-)}}{2}\right) \Psi_+^{(+)} \right] \\
&- \frac{2\beta_2}{\beta_3} \sinh(\phi_1 - \phi_2) - \frac{2\beta_3}{\beta_2} \sinh(\phi_1 + \phi_2)
\end{aligned} \tag{3.15}$$

$$\begin{aligned}
\partial_x \phi_2 - \partial_t \phi_1 = &\frac{i}{2\sqrt{2}} \zeta_1^- \left[-\beta_2 \sinh\left(\frac{\Phi_-^{(-)}}{2}\right) \bar{\Psi}_-^{(+)} - \beta_3 \sinh\left(\frac{\Phi_+^{(+)}}{2}\right) \Psi_+^{(-)} \right] \\
&+ \frac{i}{2\sqrt{2}} \frac{\beta_1}{\beta_3} \zeta_1^+ \left[-\beta_2 \sinh\left(\frac{\Phi_-^{(+)}}{2}\right) \bar{\Psi}_-^{(-)} - \beta_3 \sinh\left(\frac{\Phi_+^{(-)}}{2}\right) \Psi_+^{(+)} \right] \\
&- \frac{2\beta_2}{\beta_3} \sinh(\phi_1 - \phi_2) + \frac{2\beta_3}{\beta_2} \sinh(\phi_1 + \phi_2)
\end{aligned} \tag{3.16}$$

$$\begin{aligned}
\partial_x \varphi_1 - \partial_t \varphi_2 = &\frac{i}{2\sqrt{2}} \zeta_1^- \left[-\beta_2 \sinh\left(\frac{\Phi_-^{(-)}}{2}\right) \bar{\Psi}_-^{(+)} - \beta_3 \sinh\left(\frac{\Phi_+^{(+)}}{2}\right) \Psi_+^{(-)} \right] \\
&+ \frac{i}{2\sqrt{2}} \frac{\beta_1}{\beta_3} \zeta_1^+ \left[\beta_2 \sinh\left(\frac{\Phi_-^{(+)}}{2}\right) \bar{\Psi}_-^{(-)} + \beta_3 \sinh\left(\frac{\Phi_+^{(-)}}{2}\right) \Psi_+^{(+)} \right] \\
&- \frac{2\beta_2}{\beta_3} \sinh(\varphi_1 - \varphi_2) - \frac{2\beta_3}{\beta_2} \sinh(\varphi_1 + \varphi_2)
\end{aligned} \tag{3.17}$$

$$\partial_x \varphi_2 - \partial_t \varphi_1 = \frac{i}{2\sqrt{2}} \zeta_1^- \left[-\beta_2 \sinh\left(\frac{\Phi_-^{(-)}}{2}\right) \bar{\Psi}_-^{(+)} + \beta_3 \sinh\left(\frac{\Phi_+^{(+)}}{2}\right) \Psi_+^{(-)} \right]$$

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$$+\frac{i}{2\sqrt{2}}\frac{\beta_1}{\beta_3}\zeta_1^+\left[\beta_2\sinh\left(\frac{\Phi_-^{(+)}}{2}\right)\bar{\Psi}_-^{(-)}-\beta_3\sinh\left(\frac{\Phi_+^{(-)}}{2}\right)\Psi_+^{(+)}\right]$$
$$-\frac{2\beta_2}{\beta_3}\sinh(\varphi_1-\varphi_2)+\frac{2\beta_3}{\beta_2}\sinh(\varphi_1+\varphi_2) \tag{3.18}$$

$$\psi_1-\psi_2 = -\frac{i}{2\sqrt{2}}\beta_3\zeta_1^-\cosh\left(\frac{\Phi_+^{(+)}}{2}\right)-\frac{i}{2\sqrt{2}}\beta_1\zeta_1^+\cosh\left(\frac{\Phi_+^{(-)}}{2}\right) \tag{3.19}$$

$$\chi_1-\chi_2 = -\frac{i}{2\sqrt{2}}\beta_3\zeta_1^-\cosh\left(\frac{\Phi_+^{(+)}}{2}\right)+\frac{i}{2\sqrt{2}}\beta_1\zeta_1^+\cosh\left(\frac{\Phi_+^{(-)}}{2}\right) \tag{3.20}$$

$$\bar{\psi}_1+\bar{\psi}_2 = -\frac{i}{2\sqrt{2}}\beta_2\zeta_1^-\cosh\left(\frac{\Phi_-^{(-)}}{2}\right)-\frac{i}{2\sqrt{2}}\beta_1\frac{\beta_2}{\beta_3}\zeta_1^+\cosh\left(\frac{\Phi_-^{(+)}}{2}\right) \tag{3.21}$$

$$\bar{\chi}_1+\bar{\chi}_2 = \frac{i}{2\sqrt{2}}\beta_2\zeta_1^-\cosh\left(\frac{\Phi_-^{(-)}}{2}\right)-\frac{i}{2\sqrt{2}}\beta_1\frac{\beta_2}{\beta_3}\zeta_1^+\cosh\left(\frac{\Phi_-^{(+)}}{2}\right) \tag{3.22}$$

$$\partial_t\zeta_1^+ = -i2\sqrt{2}\frac{\beta_3}{\beta_1\beta_2}\cosh\left(\frac{\Phi_+^{(+)}}{2}\right)\Psi_+^{(-)}+i\frac{2\sqrt{2}}{\beta_1}\cosh\left(\frac{\Phi_-^{(-)}}{2}\right)\bar{\Psi}_-^{(+)} \tag{3.23}$$

$$\partial_t\zeta_1^- = -i\frac{2\sqrt{2}}{\beta_2}\cosh\left(\frac{\Phi_+^{(-)}}{2}\right)\Psi_+^{(+)}+i\frac{2\sqrt{2}}{\beta_3}\cosh\left(\frac{\Phi_-^{(+)}}{2}\right)\bar{\Psi}_-^{(-)} \tag{3.24}$$

$$\partial_x\zeta_1^+ = -i2\sqrt{2}\frac{\beta_3}{\beta_1\beta_2}\cosh\left(\frac{\Phi_+^{(+)}}{2}\right)\Psi_+^{(-)}-i\frac{2\sqrt{2}}{\beta_1}\cosh\left(\frac{\Phi_-^{(-)}}{2}\right)\bar{\Psi}_-^{(+)} \tag{3.25}$$

$$\partial_x\zeta_1^- = -i\frac{2\sqrt{2}}{\beta_2}\cosh\left(\frac{\Phi_+^{(-)}}{2}\right)\Psi_+^{(+)}-i\frac{2\sqrt{2}}{\beta_3}\cosh\left(\frac{\Phi_-^{(+)}}{2}\right)\bar{\Psi}_-^{(-)} \tag{3.26}$$

Notice that the above Backlund transformation can be re-obtained from the equations of motion (2.6) with B given by (2.15), (2.16) and (2.17)[1], that is

$$B = \frac{i}{\sqrt{2}}\zeta_1^-\left[\beta_2\cosh\left(\frac{\Phi_-^{(-)}}{2}\right)\bar{\Psi}_-^{(+)}-\beta_3\cosh\left(\frac{\Phi_+^{(+)}}{2}\right)\Psi_+^{(-)}\right]$$
$$+\frac{i}{\sqrt{2}}\frac{\beta_1}{\beta_3}\zeta_1^+\left[\beta_2\cosh\left(\frac{\Phi_-^{(+)}}{2}\right)\bar{\Psi}_-^{(-)}-\beta_3\cosh\left(\frac{\Phi_+^{(-)}}{2}\right)\Psi_+^{(+)}\right]$$
$$+\frac{2\beta_2}{\beta_3}\cosh(\phi_1-\phi_2)+\frac{2\beta_3}{\beta_2}\cosh(\phi_1+\phi_2)$$
$$-\frac{2\beta_2}{\beta_3}\cosh(\varphi_1-\varphi_2)-\frac{2\beta_3}{\beta_2}\cosh(\varphi_1+\varphi_2) \tag{3.27}$$

The above Backlund equations are also invariant under $N=2$ supersymmetry transformation (2.5). In order to obtain the one-soliton solution let us consider $\phi_2=\varphi_2=\psi_2=\chi_2\cdots=0$. Since the one-soliton solution contains only one Grassmann parameter the product of any two Fermi fields vanish and hence the Backlund equations reduce to a set

[1]The compatibility of eq. (2.6) with (3.24) require $\beta_1\beta_2=-8$

of two decoupled bosonic sinh-Gordon for ϕ_1 and φ_1. In this case we have

$$\partial_t \zeta_1^{\pm} = \lambda_-(\cosh\phi_1 + \cosh\varphi_1)\zeta_1^{\pm} \qquad \partial_x \zeta_1^{\pm} = -\lambda_+(\cosh\phi_1 + \cosh\varphi_1)\zeta_1^{\pm} \tag{3.28}$$

where $\lambda_{\pm} = \left(\frac{\beta_2}{\beta_3} \pm \frac{\beta_3}{\beta_2}\right)$. Knowing ϕ_1 and φ_1 we can integrate (3.28) to construct the auxiliary fields $\zeta_1^{\pm}$ and hence the fermionic fields $\psi_1, \chi_1, \cdots$.

The solution for ϕ_1 and φ_1 is then

$$\phi_1 = \varphi_1 = \ln\left(\frac{1+\frac{1}{2}b_1\rho_1}{1-\frac{1}{2}b_1\rho_1}\right)$$

where b_1 is an arbitrary constant and

$$\rho_1 = e^{2(\gamma_1+\gamma_1^{-1})x + 2(\gamma_1-\gamma_1^{-1})t}$$

The above solution satisfy the equation

$$(\partial_x^2 - \partial_t^2)\phi_1 = 8\sinh(2\phi_1)$$

Integrating (3.28) for $\zeta_1^{\pm}$ and parametrizing

$$\beta_3 = -\gamma_1\beta_2, \qquad \beta_2 = i2\sqrt{2}$$

we obtain the following solution

$$\begin{aligned}
\zeta_1^- &= \frac{c_1\gamma_1\rho_1}{1-\frac{1}{4}b_1^2\rho_1^2}, & \zeta_1^+ &= \gamma_1\zeta_1^- \\
\bar{\psi}_1 &= (1-\cosh\phi_1)\zeta_1^-, & \psi_1 &= \gamma_1\bar{\psi}_1 \\
\bar{\chi}_1 &= -(1+\cosh\phi_1)\zeta_1^-, & \chi_1 &= \gamma_1\bar{\chi}_1
\end{aligned}$$

where c_1 is a Grassmann constant. Using the above equations together with the eqs. of motion for the fermions we find

$$\begin{aligned}
\gamma_1(\partial_x - \partial_t)\bar{\psi}_1 &= -4\bar{\psi}_1\cosh^2\phi_1 + 4\bar{\chi}_1\sinh^2\phi_1 \\
\gamma_1(\partial_x - \partial_t)\bar{\chi}_1 &= -4\bar{\psi}_1\sinh^2\phi_1 + 4\bar{\chi}_1\cosh^2\phi_1 \\
\frac{1}{\gamma_1}(\partial_x + \partial_t)\bar{\psi}_1 &= -4\bar{\psi}_1\cosh^2\phi_1 + 4\bar{\chi}_1\sinh^2\phi_1 \\
\frac{1}{\gamma_1}(\partial_x + \partial_t)\bar{\chi}_1 &= -4\bar{\psi}_1\sinh^2\phi_1 + 4\bar{\chi}_1\cosh^2\phi_1
\end{aligned}$$

from where one can verify

$$\begin{aligned}
\gamma_1(\partial_x - \partial_t)\bar{\psi}_1 &= \frac{1}{\gamma_1}(\partial_x + \partial_t)\bar{\psi}_1 \\
\gamma_1(\partial_x - \partial_t)\bar{\chi}_1 &= \frac{1}{\gamma_1}(\partial_x + \partial_t)\bar{\chi}_1
\end{aligned}$$

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4. Zero curvature formulation

In this section we introduce the Lax pair for the $N = 2$ super sinh-Gordon model in terms of generators of the affine $sl(2,2)$ super Lie algebra (see section 3 of ref. [9] for explicit structure of generators). The Lie super algebra $sl(2,2)$ is specified by the following Bosonic, α_1, α_3 and Fermionic, α_2 simple roots

$$\alpha_1 = e_1 - e_2, \quad \alpha_3 = f_1 - f_2 \quad \text{and} \quad \alpha_2 = e_2 - f_1, \qquad e_i \cdot e_j = -f_i \cdot f_j = \delta_{ij} \tag{4.1}$$

respectively.

The Lax pair for the system described by eq. of motion (2.4) is given by

$$A_x^{(p)} = -\frac{1}{2}\partial_t\phi_p h_1 - \frac{1}{2}\partial_t\varphi_p h_3 + V_-^{(p)} \qquad A_t^{(p)} = -\frac{1}{2}\partial_x\phi_p h_1 - \frac{1}{2}\partial_x\varphi_p h_3 + V_+^{(p)} \tag{4.2}$$

where

$$\begin{aligned} V_\pm^{(p)} = & \left(-e^{-\phi_p} \pm \frac{1}{\lambda}e^{\phi_p}\right) E_{\alpha_1} + (-\lambda e^{\phi_p} \pm e^{-\phi_p})E_{-\alpha_1} \\ & +(-\lambda e^{-\varphi_p} \pm e^{\varphi_p})E_{\alpha_3} + \left(-e^{\varphi_p} \pm \frac{1}{\lambda}e^{-\varphi_p}\right) E_{-\alpha_3} + (-\lambda^{1/2} \pm \lambda^{-1/2})I \\ & +i\left[-e^{-\frac{1}{2}(\phi_p-\varphi_p)}(\psi_p + \chi_p)\lambda^{-1/4} \mp e^{\frac{1}{2}(\phi_p-\varphi_p)}(\bar\psi_p + \bar\chi_p)\lambda^{-3/4}\right] E_{\alpha_1+\alpha_2} \\ & +i\left[-e^{\frac{1}{2}(\phi_p-\varphi_p)}(\psi_p + \chi_p)\lambda^{3/4} \pm e^{-\frac{1}{2}(\phi_p-\varphi_p)}(\bar\psi_p + \bar\chi_p)\lambda^{1/4}\right] E_{-\alpha_1-\alpha_2} \\ & +i\left[e^{\frac{1}{2}(\phi_p-\varphi_p)}(\psi_p + \chi_p)\lambda^{3/4} \pm e^{-\frac{1}{2}(\phi_p-\varphi_p)}(\bar\psi_p + \bar\chi_p)\lambda^{1/4}\right] E_{\alpha_2+\alpha_3} \\ & +i\left[e^{-\frac{1}{2}(\phi_p-\varphi_p)}(\psi_p + \chi_p)\lambda^{-1/4} \mp e^{\frac{1}{2}(\phi_p-\varphi_p)}(\bar\psi_p + \bar\chi_p)\lambda^{-3/4}\right] E_{-\alpha_2-\alpha_3} \\ & +i\left[-e^{-\frac{1}{2}(\phi_p+\varphi_p)}(\psi_p - \chi_p)\lambda^{1/4} \mp e^{\frac{1}{2}(\phi_p+\varphi_p)}(\bar\psi_p - \bar\chi_p)\lambda^{-1/4}\right] E_{\alpha_1+\alpha_2+\alpha_3} \\ & +i\left[-e^{\frac{1}{2}(\phi_p+\varphi_p)}(\psi_p - \chi_p)\lambda^{1/4} \pm e^{-\frac{1}{2}(\phi_p+\varphi_p)}(\bar\psi_p - \bar\chi_p)\lambda^{-1/4}\right] E_{-\alpha_1-\alpha_2-\alpha_3} \\ & +i\left[e^{\frac{1}{2}(\phi_p+\varphi_p)}(\psi_p - \chi_p)\lambda^{1/4} \pm e^{-\frac{1}{2}(\phi_p+\varphi_p)}(\bar\psi_p - \bar\chi_p)\lambda^{-1/4}\right] E_{\alpha_2} \\ & +i\left[e^{-\frac{1}{2}(\phi_p+\varphi_p)}(\psi_p - \chi_p)\lambda^{1/4} \mp e^{\frac{1}{2}(\phi_p+\varphi_p)}(\bar\psi_p - \bar\chi_p)\lambda^{-1/4}\right] E_{-\alpha_2} \end{aligned}$$

Here $h_i = \alpha_i \cdot H$, $i = 1, 2, 3$ are the Cartan subalgebra generators, $I = h_1 + 2h_2 + h_3$ is the identity matrix and E_α denote the step operators. Notice that those step operators associated to a root α containing the simple root α_2 are fermionic in nature whilst the remaining are bosonic.

In order to describe the integrability of the system we follow [4] and split the space into two overlapping regions, namely, $x \leq b$ and $x \geq a$ with $a < b$. Inside the overlap region, i.e., $a \leq x \leq b$ introduce the following modified Lax pair

$$\begin{aligned} \hat{A}_t^{(1)} = & A_t^{(1)} + \frac{1}{2}\theta(x-a)\Big[(\partial_x\phi_1 - \partial_t\phi_2 + \partial_{\phi_1}B)h_1 + (\partial_x\varphi_1 - \partial_t\varphi_2 - \partial_{\varphi_1}B)h_3 \\ & +(\psi_1 - \psi_2 + \frac{1}{2}\partial_{\psi_1}B)E_{\alpha_2} + (\chi_1 - \chi_2 - \frac{1}{2}\partial_{\chi_1}B)E_{\alpha_2+\alpha_3} \end{aligned}$$

$$
\begin{aligned}
&+\left(\partial_t\zeta_1^+ + \frac{1}{2}\partial_{\zeta_1^-}B\right)E_{\alpha_1+\alpha_2} + (\partial_t\zeta_1^- + \frac{1}{2}\partial_{\zeta_1^+}B)E_{\alpha_1+\alpha_2+\alpha_3}\Big] \\
\hat{A}_x^{(1)} &= \theta(a-x)A_x^{(1)} \\
\hat{A}_t^{(2)} &= A_t^{(2)} + \frac{1}{2}\theta(b-x)\Big[(\partial_x\phi_2 - \partial_t\phi_1 - \partial_{\phi_2}B)h_1 + (\partial_x\varphi_2 - \partial_t\varphi_1 + \partial_{\varphi_2}B)h_3 \\
&+\left(\bar{\psi}_1 + \bar{\psi}_2 - \frac{1}{2}\partial_{\bar{\psi}_1}B\right)E_{-\alpha_2} + \left(\bar{\chi}_1 + \bar{\chi}_2 + \frac{1}{2}\partial_{\bar{\chi}_1}B\right)E_{-\alpha_2-\alpha_3} \\
&+\left(\partial_t\zeta_1^+ + \frac{1}{2}\partial_{\zeta_1^-}B\right)E_{-\alpha_1-\alpha_2} + \left(\partial_t\zeta_1^- + \frac{1}{2}\partial_{\zeta_1^+}B\right)E_{-\alpha_1-\alpha_2-\alpha_3}\Big] \\
\hat{A}_x^{(2)} &= \theta(x-b)A_x^{(2)}
\end{aligned}
$$

Within the overlap region the Lax pair denoted by suffices $p = 1, 2$ are related by gauge transformation,

$$\partial_t K = K\hat{A}_t^{(2)} - \hat{A}_t^{(1)}K.$$

Decomposing K into

$$K = e^{\frac{1}{2}\phi_2 h_1 + \frac{1}{2}\varphi_2 h_3}\bar{K}e^{-\frac{1}{2}\phi_1 h_1 - \frac{1}{2}\varphi_1 h_3} \tag{4.3}$$

we have

$$(\partial_{\phi_1}Bh_1 - \partial_{\varphi_1}Bh_3)\bar{K} + \bar{K}(\partial_{\phi_2}Bh_1 - \partial_{\varphi_2}Bh_3) = 2\bar{K}M - 2N\bar{K} - 2\partial_t\bar{K}$$

where

$$
\begin{aligned}
M &= e^{-\frac{1}{2}\phi_1 h_1 - \frac{1}{2}\varphi_1 h_3}V_+^{(2)}e^{\frac{1}{2}\phi_1 h_1 + \frac{1}{2}\varphi_1 h_3} \\
N &= e^{-\frac{1}{2}\phi_2 h_1 - \frac{1}{2}\varphi_2 h_3}V_+^{(1)}e^{\frac{1}{2}\phi_2 h_1 + \frac{1}{2}\varphi_2 h_3}
\end{aligned} \tag{4.4}
$$

which are explicitly displayed in the appendix. The solution for $\bar{K}$ is then given in the closed form

$$
\begin{aligned}
\bar{K} &= CI - \frac{\beta_3}{\beta_2}C(\lambda^{-1}E_{\alpha_1} + E_{-\alpha_1} + E_{\alpha_3} + \lambda^{-1}E_{-\alpha_3}) \\
&-\frac{C}{\beta_2}2\sqrt{2}\lambda^{-3/4}\zeta_1^+(E_{\alpha_1+\alpha_2} - \lambda E_{-\alpha_1-\alpha_2} - \lambda E_{\alpha_2+\alpha_3} + E_{-\alpha_2-\alpha_3}) \\
&+\frac{C}{2\sqrt{2}}\beta_3\lambda^{-1/4}\zeta_1^-(-E_{\alpha_2} + E_{-\alpha_2} + E_{\alpha_1+\alpha_2+\alpha_3} - E_{-\alpha_1-\alpha_2-\alpha_3})
\end{aligned} \tag{4.5}
$$

and C is an arbitrary constant. The existence of the gauge transformation (4.5) provides a generating function for an infinite set of constants of motion (see [3]) strongly indicating the integrability of the system.

The existence of Backlund transformations for bosonic and fermionic systems provide an interesting class of integrable models whose mathematical structure deserves further investigation. For instance its bi-hamiltonian properties (see for instance [14]).

The study of the quantum bosonic sinh-Gordon model with defects was explored in [12, 13]. The extension for the $N = 1$ and $N = 2$ super sinh-Gordon model with jump defects is also of interest.

These problems are under investigation.

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A. Backlund transformation for $N = 2$ super sinh-Gordon

Consider the fermionic superfields $\mathcal{F}_1$, $\mathcal{F}_2$ and $\mathcal{G}_1$, $\mathcal{G}_2$ introduced in (3.4)–(3.5) and in (3.9)–(3.10) respectively written as

$$\mathcal{F}_1 = D_+(\Xi_1^+), \qquad \mathcal{F}_2 = \bar{D}_+(\Xi_2^+) \qquad \mathcal{G}_1 = D_-(\Xi_1^-), \qquad \mathcal{G}_2 = \bar{D}_-(\Xi_2^-)$$

where

$$\Xi_1^\pm = q_1^\pm(z^\pm, \bar{z}^\pm) + \theta^\pm \zeta_1^\pm(z^\pm, \bar{z}^\pm) + \bar{\theta}^\pm \zeta_2^\pm(z^\pm, \bar{z}^\pm) + \theta^\pm \bar{\theta}^\pm q_2^\pm(z^\pm, \bar{z}^\pm)$$
$$\Xi_2^\pm = p_1^\pm(z^\pm, \bar{z}^\pm) + \theta^\pm \xi_1^\pm(z^\pm, \bar{z}^\pm) + \bar{\theta}^\pm \xi_2^\pm(z^\pm, \bar{z}^\pm) + \theta^\pm \bar{\theta}^\pm p_2^\pm(z^\pm, \bar{z}^\pm)$$

are chiral superfields. In components we find

$$\bar{D}_+ \mathcal{F}_1 = \frac{2g}{\beta_1} \sin\left(\frac{\Upsilon_1^- - \Upsilon_2^-}{2}\right)$$
$$\Downarrow$$
$$q_2^+ = \frac{2g}{\beta_1} \sin\left(\frac{\eta_-^{(-)}}{2}\right), \qquad \partial_{\bar{z}} \zeta_1^+ = \frac{g}{\beta_1} \cos\left(\frac{\eta_-^{(-)}}{2}\right) \bar{\psi}_-^{(+)}, \qquad \partial_z \zeta_2^+ = -\frac{g}{\beta_1} \cos\left(\frac{\eta_-^{(-)}}{2}\right) \psi_-^{(+)},$$
$$\partial_{\bar{z}} \partial_z q_1^+ = -\frac{g}{\beta_1} \cos\left(\frac{\eta_-^{(-)}}{2}\right) F_-^{(-)} - \frac{g}{2\beta_1} \sin\left(\frac{\eta_-^{(-)}}{2}\right) \psi_-^{(+)} \bar{\psi}_-^{(+)} \tag{A.1}$$

where we denote $\eta_\pm^{(-)} = \eta_1^- \pm \eta_2^-$, $\eta_\pm^{(+)} = \eta_1^+ \pm \eta_2^+$, similarly for the other fields.

$$D_+ \mathcal{F}_2 = -\frac{2g}{\beta_2} \sin\left(\frac{\Upsilon_1^- + \Upsilon_2^-}{2}\right)$$
$$\Downarrow$$
$$p_2^+ = \frac{2g}{\beta_2} \sin\left(\frac{\eta_+^{(-)}}{2}\right), \qquad \partial_{\bar{z}} \xi_1^+ = \frac{g}{\beta_2} \cos\left(\frac{\eta_+^{(-)}}{2}\right) \bar{\psi}_+^{(+)}, \qquad \partial_z \xi_2^+ = -\frac{g}{\beta_2} \cos\left(\frac{\eta_+^{(-)}}{2}\right) \psi_+^{(+)}$$
$$\partial_{\bar{z}} \partial_z p_1^+ = -\frac{g}{\beta_2} \cos\left(\frac{\eta_+^{(-)}}{2}\right) F_+^{(-)} - \frac{g}{2\beta_2} \sin\left(\frac{\eta_+^{(-)}}{2}\right) \psi_+^{(+)} \bar{\psi}_+^{(+)}$$

$$\bar{D}_- \mathcal{G}_1 = \frac{2g}{\beta_3} \sin\left(\frac{\Upsilon_1^+ - \Upsilon_2^+}{2}\right)$$
$$\Downarrow$$
$$q_2^- = \frac{2g}{\beta_3} \sin\left(\frac{\eta_-^{(+)}}{2}\right), \qquad \partial_{\bar{z}} \zeta_1^- = \frac{g}{\beta_3} \cos\left(\frac{\eta_-^{(+)}}{2}\right) \bar{\psi}_-^{(-)}, \qquad \partial_z \zeta_2^- = -\frac{g}{\beta_3} \cos\left(\frac{\eta_-^{(+)}}{2}\right) \psi_-^{(-)}$$
$$\partial_{\bar{z}} \partial_z q_1^- = -\frac{g}{\beta_3} \cos\left(\frac{\eta_-^{(+)}}{2}\right) F_-^{(+)} - \frac{g}{2\beta_3} \sin\left(\frac{\eta_-^{(+)}}{2}\right) \psi_-^{(-)} \bar{\psi}_-^{(-)}$$

$$D_-\mathcal{G}_2 = -\frac{2g}{\beta_4}\sin\left(\frac{\Upsilon_1^+ + \Upsilon_2^+}{2}\right)$$

$$\Downarrow$$

$$p_2^- = \frac{2g}{\beta_4}\sin\left(\frac{\eta_+^{(+)}}{2}\right), \qquad \partial_{\bar{z}}\xi_1^- = \frac{g}{\beta_4}\cos\left(\frac{\eta_+^{(+)}}{2}\right)\bar{\psi}_+^{(-)}, \qquad \partial_z\xi_2^- = -\frac{g}{\beta_4}\cos\left(\frac{\eta_+^{(+)}}{2}\right)\psi_+^{(-)}$$

$$\partial_{\bar{z}}\partial_z p_1^- = -\frac{g}{\beta_4}\cos\left(\frac{\eta_+^{(+)}}{2}\right)F_+^{(+)} - \frac{g}{2\beta_4}\sin\left(\frac{\eta_+^{(+)}}{2}\right)\psi_+^{(-)}\bar{\psi}_+^{(-)} \tag{A.2}$$

$$D_+\Upsilon_1^+ = D_+\Upsilon_2^+ + \beta_1\mathcal{F}_1\cos\left(\frac{\Upsilon_1^- + \Upsilon_2^-}{2}\right)$$

$$\Downarrow$$

$$\psi_-^{(-)} = \beta_1\zeta_1^+\cos\left(\frac{\eta_+^{(-)}}{2}\right), \qquad \partial_z\eta_-^{(+)} = \frac{\beta_1}{2}\sin\left(\frac{\eta_+^{(-)}}{2}\right)\zeta_1^+\psi_+^{(+)} + \beta_1\partial_z q_1^+\cos\left(\frac{\eta_+^{(-)}}{2}\right)$$

$$F_-^{(+)} = \beta_1 q_2^+\cos\left(\frac{\eta_+^{(-)}}{2}\right)$$

$$\bar{D}_+\Upsilon_1^+ = -\bar{D}_+\Upsilon_2^+ + \beta_2\mathcal{F}_2\cos\left(\frac{\Upsilon_1^- - \Upsilon_2^-}{2}\right)$$

$$\Downarrow$$

$$\bar{\psi}_+^{(-)} = \beta_2\xi_2^+\cos\left(\frac{\eta_-^{(-)}}{2}\right), \qquad \partial_{\bar{z}}\eta_+^{(+)} = \frac{\beta_2}{2}\sin\left(\frac{\eta_-^{(-)}}{2}\right)\xi_2^+\bar{\psi}_-^{(+)} + \beta_2\partial_{\bar{z}} p_1^+\cos\left(\frac{\eta_-^{(-)}}{2}\right)$$

$$F_+^{(+)} = \beta_2 p_2^+\cos\left(\frac{\eta_-^{(-)}}{2}\right)$$

$$D_-\Upsilon_1^- = D_-\Upsilon_2^- + \beta_3\mathcal{G}_1\cos\left(\frac{\Upsilon_1^+ + \Upsilon_2^+}{2}\right)$$

$$\Downarrow$$

$$\psi_-^{(+)} = \beta_3\zeta_1^-\cos\left(\frac{\eta_+^{(+)}}{2}\right), \qquad \partial_z\eta_-^{(-)} = \frac{\beta_3}{2}\sin\left(\frac{\eta_+^{(+)}}{2}\right)\zeta_1^-\psi_+^{(-)} + \beta_3\partial_z q_1^-\cos\left(\frac{\eta_+^{(+)}}{2}\right)$$

$$F_-^{(-)} = \beta_3 q_2^-\cos\left(\frac{\eta_+^{(+)}}{2}\right)$$

$$\bar{D}_{-}\Upsilon_1^{-} = -\bar{D}_{-}\Upsilon_2^{-} + \beta_4 \mathcal{G}_2 \cos\left(\frac{\Upsilon_1^{+} - \Upsilon_2^{+}}{2}\right)$$

$$\Downarrow$$

$$\bar{\psi}_{+}^{(+)} = \beta_4 \xi_2^{-} \cos\left(\frac{\eta_{-}^{(+)}}{2}\right), \qquad \partial_{\bar{z}}\eta_{+}^{(-)} = \frac{\beta_4}{2}\sin\left(\frac{\eta_{-}^{(+)}}{2}\right)\xi_2^{-}\bar{\psi}_{-}^{(-)} + \beta_4 \partial_{\bar{z}} p_1^{-} \cos\left(\frac{\eta_{-}^{(+)}}{2}\right)$$

$$F_{+}^{(-)} = \beta_4 p_2^{-} \cos\left(\frac{\eta_{-}^{(+)}}{2}\right)$$

Acting $\bar{D}_{-}$ in eq. (3.4) and using eq. (3.10), we obtain

$$\mathcal{F}_1 \mathcal{G}_2 = 0$$

which is satisfied when

$$\zeta_1^{+} = \xi_2^{-}, \qquad q_2^{+} = \partial_{\bar{z}} p_1^{-}, \qquad p_2^{-} = -\partial_z q_1^{+}, \qquad \partial_z \zeta_2^{+} = -\partial_{\bar{z}} \xi_1^{-} \tag{A.3}$$

Similarly, acting with D_{-} in eq. (3.5) and making use of eq. (3.9),we obtain

$$\mathcal{F}_2 \mathcal{G}_1 = 0$$

which is satisfied when

$$\zeta_1^{-} = \xi_2^{+}, \qquad p_2^{+} = -\partial_z q_1^{-}, \qquad q_2^{-} = \partial_{\bar{z}} p_1^{+}, \qquad \partial_z \zeta_2^{-} = -\partial_{\bar{z}} \xi_1^{+} \tag{A.4}$$

Making use of eq. (A.3) and (A.1), we find

$$\partial_{\bar{z}}\partial_z q_1^{+} = -\partial_{\bar{z}} p_2^{-} = -\frac{g}{\beta_1}\cos\left(\frac{\eta_{-}^{(-)}}{2}\right) F_{-}^{(-)} - \frac{g}{2\beta_1}\sin\left(\frac{\eta_{-}^{(-)}}{2}\right)\psi_{-}^{(+)}\bar{\psi}_{-}^{(+)} \tag{A.5}$$

Acting $\partial_{\bar{z}}$ in the first eq. (A.2) we obtain

$$\partial_{\bar{z}} p_2^{-} = \frac{2g}{\beta_4}\partial_{\bar{z}}\left[\sin\left(\frac{\eta_{+}^{(+)}}{2}\right)\right] \tag{A.6}$$

In order to (A.5) be compatible with (A.6), it is necessary that

$$\beta_1 \beta_2 = \beta_3 \beta_4 \tag{A.7}$$

B. Explicit expressions for M and N

Here we give detailed expression for M and N of eq. (4.4)

$$\begin{aligned} M &= a_{-}E_{\alpha_1} + b_{+}E_{-\alpha_1} + c_{-}E_{\alpha_3} + d_{+}E_{-\alpha_3} + \lambda_{+} I \\ &\quad -\alpha_{+}^{(2)} E_{\alpha_1+\alpha_2} - \beta_{-}^{(2)} E_{-\alpha_1-\alpha_2} + \beta_{+}^{(2)} E_{\alpha_2+\alpha_3} + \alpha_{-}^{(2)} E_{-\alpha_2-\alpha_3} \\ &\quad -\gamma_{+}^{(2)} E_{\alpha_1+\alpha_2+\alpha_3} - \delta_{-}^{(2)} E_{-\alpha_1-\alpha_2-\alpha_3} + \delta_{+}^{(2)} E_{\alpha_2} + \gamma_{-}^{(2)} E_{-\alpha_2} \\ N &= a_{+}E_{\alpha_1} + b_{-}E_{-\alpha_1} + c_{+}E_{\alpha_3} + d_{-}E_{-\alpha_3} + \lambda_{+} I \end{aligned}$$

$$
\begin{aligned}
&-\alpha_+^{(1)} E_{\alpha_1+\alpha_2} - \beta_-^{(1)} E_{-\alpha_1-\alpha_2} + \beta_+^{(1)} E_{\alpha_2+\alpha_3} + \alpha_-^{(1)} E_{-\alpha_2-\alpha_3} \\
&-\gamma_+^{(1)} E_{\alpha_1+\alpha_2+\alpha_3} - \delta_-^{(1)} E_{-\alpha_1-\alpha_2-\alpha_3} + \delta_+^{(1)} E_{\alpha_2} + \gamma_-^{(1)} E_{-\alpha_2}
\end{aligned}
$$

where

$$
\begin{aligned}
a_- &= (-e^{-\phi_+} + \lambda^{-1} e^{-\phi_-}), \qquad b_+ = (-\lambda e^{\phi_+} + e^{\phi_-}), \\
c_- &= (-\lambda e^{-\varphi_+} + e^{-\varphi_-}), \qquad d_+ = (-e^{\varphi_+} + \lambda^{-1} e^{\varphi_-}), \\
\alpha_\pm^{(2)} &= i \left[e^{-\frac{1}{2}\phi_+^{(-)}} (\psi_2 + \chi_2) \lambda^{-1/4} \pm e^{-\frac{1}{2}\phi_-^{(-)}} (\bar{\psi}_2 + \bar{\chi}_2) \lambda^{-3/4} \right] \\
\beta_\pm^{(2)} &= i \left[e^{\frac{1}{2}\phi_+^{(-)}} (\psi_2 + \chi_2) \lambda^{3/4} \pm e^{\frac{1}{2}\phi_-^{(-)}} (\bar{\psi}_2 + \bar{\chi}_2) \lambda^{1/4} \right] \\
\gamma_\pm^{(2)} &= i \left[e^{-\frac{1}{2}\phi_+^{(+)}} (\psi_2 - \chi_2) \lambda^{1/4} \pm e^{-\frac{1}{2}\phi_-^{(+)}} (\bar{\psi}_2 - \bar{\chi}_2) \lambda^{-1/4} \right] \\
\delta_\pm^{(2)} &= i \left[e^{\frac{1}{2}\phi_+^{(+)}} (\psi_2 - \chi_2) \lambda^{1/4} \pm e^{\frac{1}{2}\phi_-^{(+)}} (\bar{\psi}_2 - \bar{\chi}_2) \lambda^{-1/4} \right] \\
\lambda_\pm &= (-\lambda^{1/2} \pm \lambda^{-1/2})
\end{aligned}
$$

and

$$
\begin{aligned}
a_+ &= (-e^{-\phi_+} + \lambda^{-1} e^{\phi_-}), \qquad b_- = (-\lambda e^{\phi_+} + e^{-\phi_-}) \\
c_+ &= (-\lambda e^{-\varphi_+} + e^{\varphi_-}), \qquad d_- = (-e^{\varphi_+} + \lambda^{-1} e^{-\varphi_-}), \\
\alpha_\pm^{(1)} &= i \left[e^{-\frac{1}{2}\phi_+^{(-)}} (\psi_1 + \chi_1) \lambda^{-1/4} \pm e^{\frac{1}{2}\phi_-^{(-)}} (\bar{\psi}_1 + \bar{\chi}_1) \lambda^{-3/4} \right] \\
\beta_\pm^{(1)} &= i \left[e^{\frac{1}{2}\phi_+^{(-)}} (\psi_1 + \chi_1) \lambda^{3/4} \pm e^{-\frac{1}{2}\phi_-^{(-)}} (\bar{\psi}_1 + \bar{\chi}_1) \lambda^{1/4} \right] \\
\gamma_\pm^{(1)} &= i \left[e^{-\frac{1}{2}\phi_+^{(+)}} (\psi_1 - \chi_1) \lambda^{1/4} \pm e^{\frac{1}{2}\phi_-^{(+)}} (\bar{\psi}_1 - \bar{\chi}_1) \lambda^{-1/4} \right] \\
\delta_\pm^{(1)} &= i \left[e^{\frac{1}{2}\phi_+^{(+)}} (\psi_1 - \chi_1) \lambda^{1/4} \pm e^{-\frac{1}{2}\phi_-^{(+)}} (\bar{\psi}_1 - \bar{\chi}_1) \lambda^{-1/4} \right]
\end{aligned}
$$

Acknowledgments

LHY acknowledges support from Fapesp, JFG and AHZ thank CNPq for partial support.

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